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FREQUENCY AND BUCKLING EIGENVALUES OF ANISOTROPIC CYLINDERS SUBJECTED TO NONUNIFORM LATERAL PRESTRESS

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Abstract—Solutions are developed herein for the evaluation of frequency and stability eigenvalues of macroscopically fully anisotropic circular cylindrical shells subjected to nonuniform lateral prestress. Included in the analysis is the presence of torsional prestress. The results obtained are applicable to any type of prestress which satisfy Dirichlet's conditions for Fourier series. Furthermore, in the manner of Kalnins [3], using the procedure outlined herein, similar results could be obtained for general anisotropic shells of revolution.

INTRODUCTION

IN RECENT years with the increased usage of composite materials in structures, an understanding of the effects of anisotropy on shell characteristics is of increased importance. In this respect, although recent analyses [1–4] have been made on the lateral stability of cylindrical [1, 2, 4] and general shells of revolution [3], to date, these have been restricted solely to orthotropic constitutive equations. This paper develops solutions for obtaining the stability and frequency eigenvalues of macroscopically anisotropic circular cylindrical shells subjected to nonuniform lateral prestress. The results obtained are applicable to any type of prestress which can be expanded in Fourier series. To extend the scope of the present analysis, torsional and "moderate" axial prestresses are also included in the development. As the solutions derived herein are general, setting the anisotropic elastic compliances to zero yields the results of previous investigators [1, 2, 4] as special cases.

SHELL GEOMETRY

The position of a point of a circular cylindrical shell is given by θ the circumferential distance, x the axial distance and z the coordinate normal to the middle surface. For the present study, R is the radius of the cylinder measured from the axis of rotation to the middle surface and L is the length of the shell. The displacement of the middle surface in the circumferential, axial and outward radial directions are denoted by u, v and w respectively.

EQUATIONS

The present analysis is based on Flugge [5] shell theory with Hoppmann [6, 7] type anisotropic constitutive relations being used to derive the shell stability equations of

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motion. For completeness the shell constitutive relations are given by

$$\begin{pmatrix} \sigma_x^{(0,1)} \\ \sigma_\theta^{(0,1)} \\ \sigma_{x\theta}^{(0,1)} \\ \sigma_{\theta x}^{(0,1)} \end{pmatrix} = \begin{pmatrix} E_{11}^{(0,1)} & E_{12}^{(0,1)} & E_{13}^{(0,1)} & 0 \\ E_{12}^{(0,1)} & E_{22}^{(0,1)} & 0 & E_{13}^{(0,1)} \\ E_{13}^{(0,1)} & E_{23}^{(0,1)} & E_{33}^{(0,1)} & 0 \\ E_{13}^{(0,1)} & E_{23}^{(0,1)} & 0 & E_{33}^{(0,1)} \end{pmatrix} \begin{pmatrix} \varepsilon_x^{(0,1)} \\ \varepsilon_\theta^{(0,1)} \\ \varepsilon_{x\theta}^{(0,1)} \\ \varepsilon_{\theta x}^{(0,1)} \end{pmatrix} \tag{1}$$

where $^{(0)}$ and $^{(1)}$ superscripts denote inplane and bending shell properties. The terms $\varepsilon_x^{(0,1)}, \varepsilon_\theta^{(0,1)}, \ldots$ are the integrated shell strains defined by

$$\begin{aligned} \varepsilon_x^{(0,1)} &= \int_{-h_p/2,h_b/2}^{h_p/2,h_b/2} \varepsilon_x(1,z)\left(\frac{R+z}{R}\right) \mathrm{d}z \\ \varepsilon_\theta^{(0,1)} &= \ldots \\ \varepsilon_{x\theta}^{(0,1)} &= \ldots \\ \varepsilon_{\theta x}^{(0,1)} &= \int_{-h_p/2,h_b/2}^{h_p/2,h_b/2} \varepsilon_{\theta x}(1,z)\,\mathrm{d}z \end{aligned} \tag{2}$$

with $\varepsilon_x, \ldots$ being the usual Flugge [5] shell strains, and h_p, h_b being the effective inplane and bending shell thicknesses. Using equations (1) the shell stability equations of motions are

$$a_1 u_{,xx} + a_2 u_{,\theta x} + a_3 u_{,\theta\theta} + a_4 v_{,xx} + a_5 v_{,\theta x} + a_6 v_{,\theta\theta} + a_7 w_{,x} + a_8 w_{,\theta} + a_9 w_{,\theta\theta x} + a_{10} w_{,xx\theta}$$
$$+ a_{11} w_{,xxx} + a_{12} w_{,\theta\theta\theta} - \rho R^2 h u_{,tt} = L_1(u, v, w, \ldots, x, \theta, t) = 0 \tag{3a}$$

$$b_1 u_{,xx} + b_2 u_{,\theta\theta} + b_3 u_{,x\theta} + b_4 v_{,xx} + b_5 v_{,\theta\theta} + b_6 v_{,x\theta} + b_7 w_{,xxx} + b_8 w_{,\theta\theta\theta} + b_9 w_{,x\theta\theta} + b_{10} w_{,xx\theta}$$
$$+ b_{11} w_{,x} + b_{12} w_{,\theta} - \rho R^2 h v_{,tt} = L_2(u, v, \ldots, x, \theta, t) = 0 \tag{3b}$$

$$c_1 u_{,xxx} + c_2 u_{,\theta\theta\theta} + c_3 u_{,\theta xx} + c_4 u_{,\theta\theta x} + c_5 u_{,x} + c_6 u_{,\theta} + c_7 v_{,xxx} + c_8 v_{,x\theta\theta} + c_9 v_{,xx\theta} + c_{10} v_{,x}$$
$$+ c_{11} v_{,\theta} + c_{12} w_{,xxxx} + c_{13} w_{,\theta\theta\theta\theta} + c_{14} w_{,\theta\theta xx} + c_{15} w_{,\theta\theta\theta x} + c_{16} w_{,xxx\theta} + c_{17} w_{,xx}$$
$$+ c_{18} w_{,\theta\theta} + c_{19} w_{,x\theta} + c_{20} w + \rho R^2 h w_{,tt} = L_3(u, v, \ldots, x, \theta, t) = 0 \tag{3c}$$

where h is the effective shell thickness associated with the inertia terms, with $a_1, a_2, \ldots, a_{12}, b_1, b_2, \ldots, b_{12}$ and $c_1, c_2, c_3, \ldots, c_{20}$ defined in Appendix 1. The membrane prestresses P_a (axial), P_r (lateral) and $\bar{T}$ (torque) enter equations (3a)–(3c) through the definitions given for a_i, b_i, c_l; $i = 1, 2, \ldots, 12$, $l = 1, 2, 3, \ldots, 20$.

SOLUTION

In physical situations where the lateral pressure P_r is not uniform, equations (3) are partials with variable coefficients. For this case the usual Fourier decomposition [8, 9] in θ and exponential substitutions [10] in x are not directly applicable. In the early 1960's Almroth [1] obtained an approximate solution for the case of buckling of isotropic cylinders due to nonuniform lateral pressure loads. Using the Rayleigh–Ritz procedure he expanded the displacement components in trigonometric series and obtained a solution for freely supported cylindrical shells with loadings of the type

$$P_r \propto P_0(1+\varepsilon\cos\theta). \tag{4}$$

An alternate solution procedure was used by Hoff [2], Kempner *et al.* [11], Kalnins [3] and Samuelson [4] for their respective studies. This method consists of expanding the displacement components into Fourier cosine and sine series. These infinite series are substituted into the governing differential equations. The resulting ∞ set of equations is then truncated and solved for the required information. The preceding method of solution is not directly applicable since the typical solution characterized by the usual Fourier decomposition procedure cannot satisfy the present governing differential equations. This is due to the presence of the anisotropic compliances ($E_{13}^{(0,1)}$, $E_{23}^{(0,1)}$) and torque prestress $\bar{T}$. For anisotropic cylinders with uniform prestresses, equations (3a)–(3c) can be reduced to one of the canonical forms with the aid of appropriate transformations. Although such transformations exist, the associated change in the domain of definition distorts the bounding surfaces of the cylinder. Therefore to circumvent these difficulties, the procedure discussed by Padovan [12] will be used to construct a solution.

To simplify the present discussion $P_r(x,\theta)$ is expanded in Fourier series, i.e.

$$P_r(\theta) = \sum_{\gamma=0}^{\infty}[\zeta_\gamma(x)\cos\gamma\theta+\Gamma_\gamma(x)\sin\gamma\theta] \tag{5}$$

where convergence is assured for loadings which satisfy Dirichlet's conditions. Now applying the usual finite Fourier exponential transform to equations (3), assuming that $\int\sum\to\sum\int$, we have

$$\int_0^{2\pi}\begin{pmatrix}L_1(u,v,w,\ldots,x,\theta,t)\\ L_2(u,v,w,\ldots,x,\theta,t)\\ L_3(u,v,w,\ldots,x,\theta,t)\end{pmatrix} e^{-jM\theta}\,d\theta\to$$

$$\begin{aligned}&a_1T\{u_{,xx}\}+a_2T\{u_{,\theta x}\}+RE_{33}^{(0)}[k+k_p]T\{u_{,\theta\theta}\}+a_4T\{v_{,xx}\}\\&\quad+a_5T\{v_{,\theta x}\}+a_6T\{v_{,\theta\theta}\}+RE_{12}^{(0)}kT\{w_{,x}\}+a_8T\{w_{,\theta}\}\\&\quad+a_9T\{w_{,\theta\theta x}\}+a_{10}T\{w_{,\theta xx}\}+a_{11}T\{w_{,xxx}\}\\&\quad+a_{12}T\{w_{,\theta\theta\theta}\}-\rho R^2hT\{u_{,tt}\}-P_0R\sum_{\gamma=0}^{\infty}[T\{\zeta_\gamma(x)(u_{,\theta\theta}-w_{,x})\cos\gamma\theta\}\\&\quad+T\{\Gamma_\gamma(x)(u_{,\theta\theta}-w_{,x})\sin\gamma\theta\}]=\mathbf{L}_1(u_c,u_s,\ldots,x,M,t)=0.\end{aligned} \tag{6a}$$

$$\begin{aligned}&b_1T\{u_{,xx}\}+b_2T\{u_{,\theta\theta}\}+b_3T\{u_{,\theta x}\}+b_4T\{v_{,xx}\}\\&\quad+RkE_{22}^{(0)}T\{v_{,\theta\theta}\}+b_6T\{v_{,\theta x}\}+b_7T\{w_{,xxx}\}+b_8T\{w_{,\theta\theta\theta}\}\\&\quad+b_9T\{w_{,\theta\theta x}\}+b_{10}T\{w_{,\theta xx}\}+b_{11}T\{w_{,x}\}+[RE_{22}^{(0)}(k+k_p)\\&\quad-E_{22}^{(1)}k_b]T\{w_{,\theta}\}-\rho R^2hT\{v_{,tt}\}-P_0R\sum_{\gamma=0}^{\infty}[T\{\zeta_\gamma(x)(v_{,\theta\theta}+w_{,\theta})\cos\gamma\theta\}\\&\quad+T\{\Gamma_\gamma(x)(v_{,\theta\theta}+w_{,\theta})\sin\gamma\theta\}]=\mathbf{L}_2(u_c,u_s,\ldots,x,M,t)=0.\end{aligned} \tag{6b}$$

$$
\begin{aligned}
& c_1 T\{u_{,xxx}\}+c_2 T\{u_{,\theta\theta\theta}\}+c_3 T\{u_{,\theta xx}\}+c_4 T\{u_{,\theta\theta x}\} \\
& \quad + RE_{12}^{(0)} k T\{u_{,x}\}+c_6 T\{u_{,\theta}\}+c_7 T\{v_{,xxx}\}+c_8 T\{v_{,\theta\theta x}\} \\
& \quad + c_9 T\{v_{,xx\theta}\}+c_{10} T\{v_{,x}\}+RE_{22}^{(0)} T\{v_{,\theta}\}+c_{12} T\{w_{,xxxx}\} \\
& \quad + c_{13} T\{w_{,\theta\theta\theta\theta}\}+c_{14} T\{w_{,\theta\theta xx}\}+c_{15} T\{w_{,\theta\theta\theta x}\}+c_{16} T\{w_{,\theta xxx}\} \\
& \quad + c_{17} T\{w_{,xx}\}+(RE_{22}^{(0)} k_p+E_{22}^{(1)} k_b) T\{w_{,\theta\theta}\}+c_{19} T\{w_{,\theta x}\} \\
& \quad + c_{20} T\{w\}+\rho R^2 h T\{w_{,tt}\}+P_0 R \sum_{\gamma=0}^{\infty} [T\{\zeta_\gamma(x)(u_{,x}-v_{,\theta}+w_{,\theta\theta}) \cos \gamma\theta\} \\
& \quad + T\{\Gamma_\gamma(x)(u_{,x}-v_{,\theta}+w_{,\theta\theta}) \sin \gamma\theta\}] = \mathbf{L}_3(u_c, u_s, \ldots, x, M, t) = 0. \qquad (6c)
\end{aligned}
$$

where $j = \sqrt{(-1)}$ and $T\{u\}, \ldots$ denote

$$
\begin{aligned}
T\{u\} &\rightarrow u_c(x, M, t)-j u_s(x, M, t) \\
T\{u_{,\theta}\} &\rightarrow M[u_s(x, M, t)+j u_c(x, M, t)] \\
T\{u_{,\theta\theta}\} &\rightarrow M^2[-u_c(x, M, t)+j u_s(x, M, t)] \\
T\{u \cos \gamma\theta\} &\rightarrow \tfrac{1}{2}[u_c(x, M-\gamma, t)+u_c(x, M+\gamma, t)-j(u_s(x, M-\gamma, t)+u_s(x, M+\gamma, t))] \qquad (7a) \\
&\vdots
\end{aligned}
$$

with $u_c(x, M \pm \gamma, t), u_s(x, M \pm \gamma, t), \ldots$ being finite cosine and sine transforms of $u(x, \theta, t)$, $v(x, \theta, t)$ and $w(x, \theta, t)$ respectively. The functional dependence of $u_c(x, M \pm \gamma, t), \ldots$ with respect to M and γ is given by:

$$
\begin{pmatrix} u_c(x, M \pm \gamma, t), u_s(x, M \pm \gamma, t) \\ v_c(x, M \pm \gamma, t), v_s(x, M \pm \gamma, t) \\ w_c(x, M \pm \gamma, t), w_s(x, M \pm \gamma, t) \end{pmatrix} = \int_0^{2\pi} \begin{pmatrix} u(x, \theta, t) \\ v(x, \theta, t) \\ w(x, \theta, t) \end{pmatrix} (\cos (M \pm \gamma) \theta, \sin (M \pm \gamma)\theta) \, d\theta. \qquad (7b)
$$

In the manner of Padovan [12], as the real and imaginary parts of the exponential transform form linearly independent basis spaces, equations (7a) reduce to

$$
\begin{aligned}
T\{u\} &\rightarrow [u_c(x, M, t), u_s(x, M, t)] \\
T\{u_{,\theta}\} &\rightarrow M[u_s(x, M, t), u_c(x, M, t)] \\
T\{u_{,\theta\theta}\} &\rightarrow M^2[-u_c(x, M, t), u_s(x, M, t)] \\
T\{u \cos \gamma\theta\} &\rightarrow \tfrac{1}{2}[u_c(x, M-\gamma, t)+u_c(x, M+\gamma, t), -u_s(x, M-\gamma, t)-u_s(x, M+\gamma, t)] \qquad (7c) \\
&\vdots
\end{aligned}
$$

With the new definitions of (7c), equations (6a)–(6c) constitute two coupled infinite sets of ordinary differential equations, both sets having as dependent variables $u_c(x, M, t)$, $u_c(x, M \pm \gamma, t)$, $u_s(x, M, t)$, $u_s(x, M \pm \gamma, t), \ldots, w_s(x, M \pm \gamma, t)$. Setting the elastic compliances $E_{13}^{(0,1)}$ and $E_{23}^{(0,1)}$ to zero reduces equations (6) into two independent subsets with dependent variables $u_c(x, M, t)$, $u_c(x, M \pm \gamma, t)$, $v_s(x, M, t)$, $v_s(x, M \pm \gamma, t)$, $w_c(x, M, t)$, $w_c(x, M \pm \gamma, t)$ and $u_s(x, M, t), \ldots, w_s(x, M \pm \gamma, t)$, respectively. These resultant equations are applicable to the usual orthotropic and isotropic cases.

To obtain the required frequency eigenvalue relations from equations (6), we assume as usual that

$$\{u(x, \theta, t), v(x, \theta, t), w(x, \theta, t)\} \propto \{U(x, \theta), V(x, \theta), W(x, \theta)\} e^{j\omega t}. \tag{8}$$

Substituting (8) into (6) yields the frequency relations. The general solution of these relations for the stability criteria and frequency eigenvalues of anisotropic cylinders under arbitrary $P_r(x, \theta)$ and boundary conditions presents a large scale numerical problem. For the present investigations two special cases are considered in detail. Before discussing these, it is noted that recently Kalnins [3] developed a plausible numerical solution and resultant computer program for linear stability problems in orthotropic shells of revolution. Included in his [3] analysis was the capability of analyzing lateral bucking due to nonuniform prestresses. Using the procedure outlined herein, Kalnins [3] results can be directly extended to anisotropic shells of revolution.

ORTHOTROPIC AND ANISOTROPIC CYLINDERS WITH ARBITRARY BOUNDARY CONDITIONS

A formal solution to equations (6a)–(6c) for arbitrary boundary conditions can be developed for the special class of functions $Pr(\theta)$. For this case it is noted that as usual

$$(U_c(x, M), U_s(x, M), \ldots) \propto (\bar{U}_c(M), \bar{U}_s(M), \ldots) e^{\lambda x} \tag{9}$$

where here λ is an unknown. Using such a solution form, the frequency and buckling eigenvalues may be obtained in a manner similar to that used by Forsberg [10]. For the present equations, substituting (9) into (6) yields an ∞ matrix equation of the form

$$\left\{\sum_{i=0}^{4} A_{4-i}\lambda^i\right\}\psi = D(\lambda)\psi = 0 \tag{10}$$

where $A_i(P_0, \omega, \ldots)$ $i = 0, 1, \ldots, 4$ are infinite matrices and the transpose of ψ is given by

$$(\bar{U}_c(0), \bar{W}_c(0), \bar{U}_c(1), \bar{V}_s(1), \ldots) = \psi'. \tag{11}$$

In its present form, since A_0 is singular, the pencil of equation (10) is an irregular polynomial matrix. Noting that A_4 is nonsingular, $D(\lambda)$ can be transformed into a regular polynomial matrix. This is done through the transformation

$$\psi = \frac{1}{\lambda^4}\boldsymbol{\psi} \tag{12}$$

where here it is assumed that $\lambda \neq 0$. Thus we have that

$$\left\{\sum_{i=0}^{4} A_i\boldsymbol{\lambda}^i\right\}\boldsymbol{\psi} = D(\boldsymbol{\lambda})\boldsymbol{\psi} = 0 \tag{13}$$

where

$$\boldsymbol{\lambda} = \frac{1}{\lambda}. \tag{14}$$

To obtain a solution to (13), A_i; $i = 0, 1, \ldots, 4$ are truncated to finite size. Even in its truncated form a direct solution to (13) is quite difficult. To alleviate this situation, a basic property of polynomial matrices will be used. That is, since the pencil $\mathbf{D}(\lambda)$ is a regular polynomial matrix, then equation (13) implies and is implied by the partitioned matrix equation

$$\left[\begin{pmatrix} 0 & 0 & 0 & A_4 \\ 0 & 0 & A_4 & A_3 \\ 0 & A_4 & A_3 & A_2 \\ A_4 & A_3 & A_2 & A_1 \end{pmatrix}\lambda + \begin{pmatrix} 0 & 0 & -A_4 & 0 \\ 0 & -A_4 & -A_3 & 0 \\ -A_4 & -A_3 & -A_2 & 0 \\ 0 & 0 & 0 & A_0 \end{pmatrix}\right]\begin{pmatrix} \psi^{(3)} \\ \psi^{(2)} \\ \psi^{(1)} \\ \psi \end{pmatrix} = 0 \tag{15}$$

or more simply

$$\{R_0\lambda + R_1\}\mu = 0. \tag{16}$$

Since A_4 is nonsingular, (16) can be written as

$$\{R_0^{-1}R_1 + [I]\lambda\}\mu = 0. \tag{17}$$

Equation (17) denotes the typically occuring linear eigenvalue problem.

To implement the procedure discussed by Forsberg [10] the transformed boundary conditions at $x = 0, L$ are required. In general, the shell *b.c.* at $x = 0, L$ are given by

$$\begin{aligned} \sigma_x^{(0)} + k_x u &= 0 \\ \sigma_x^{(0)} - \frac{1}{R}\sigma_{x\theta}^{(1)} + k_\theta v &= 0 \\ \sigma_{x,x}^{(1)} + \frac{1}{R}(\sigma_{\theta x,\theta}^{(1)} + \sigma_{x\theta,\theta}^{(1)}) + k_z w &= 0 \\ \sigma_x^{(1)} + k_{x\theta} w_{,x} &= 0 \end{aligned} \tag{18}$$

where here k_x, k_θ, k_z and $k_{x\theta}$ denote spring constants at the shell edges. Applying the transform discussed earlier we have

$$\begin{aligned} \sigma_{\substack{c\\s}x}^{(0)} + k_x U_{\substack{c\\s}} &= 0 \\ \sigma_{\substack{c\\s}x\theta}^{(0)} - \frac{1}{R}\sigma_{\substack{c\\s}x\theta}^{(1)} + k_\theta V_{\substack{c\\s}} &= 0 \\ \sigma_{\substack{c\\s}x,x}^{(1)} \pm \frac{M}{R}(\sigma_{\substack{s\\c}\theta x}^{(1)} + \sigma_{\substack{s\\c}x\theta}^{(1)}) + k_z W_{\substack{c\\s}} &= 0 \\ \sigma_{\substack{c\\s}x}^{(1)} + k_{x\theta} W_{\substack{c\\s},x} &= 0. \end{aligned} \tag{19}$$

In the manner of Forsberg [10], the λ of (17) may be obtained for assumed values of ω and/or P_0. These are used to construct $U_c(x, M)$, $U_s(x, M), \ldots$, i.e.

$$\begin{aligned} U_{\substack{c\\s}}(x, M) &= \sum_{i=1}^{\xi} \bar{U}_{\substack{c\\s}}(M)\, e^{x/\lambda i} \\ V_{\substack{c\\s}}(x, M) &= \sum_{i=1}^{\xi} \bar{V}_{\substack{c\\s}}(M)\, e^{x/\lambda i} \\ W_{\substack{c\\s}}(x, M) &= \sum_{i=1}^{\xi} \bar{W}_{\substack{c\\s}}(M)\, e^{x/\lambda i} \end{aligned} \tag{20}$$

where ξ is dependent on the truncated size of (13). From this point, the usual procedure follows [10].

In general, the matrix $R_0^{-1}R_1$ is unbanded and non-Hermitian. For the special case of loadings of the form

$$P_r(\theta) = P_0\left\{1 + \Delta \sum_{\gamma=0}^{\infty} P_\gamma \cos \gamma\theta\right\}$$
$$P_\gamma < 1, \Delta \ll 1 \tag{21}$$

$R_0^{-1}R_1$ is well conditioned since the diagonal band, consisting of $\gamma = 0$ terms, is dominant. Furthermore it can be shown that the terms corresponding to the critical buckling mode of the uniform case are predominant. Therefore the λ and P_0 associated with the uniform case can be used as starting values of the iteration procedure.

As the magnitude of Δ in equation (21) increases, the importance of the terms associated with $\gamma > 0$ is amplified. The resultant effect is an increase in the size of $R_0^{-1}R_1$ required for adequate convergence. This is particularly true for slowly convergent Fourier series representing $P_r(\theta)$.

For the special case of uniform external pressure, (10) reduces to a finite matrix equation, in which, ψ is given by

$$\psi' = (\bar{U}_c, \bar{V}_s, \bar{W}_c, \bar{U}_s, \bar{V}_c, \bar{W}_s). \tag{22}$$

The form of equation (10) can be further simplified by observing that making the appropriate additions and subtractions we have

$$\sum_{i=0}^{4}\left[\begin{pmatrix} \Lambda^{(i)}_{(I,J)} & 0 \\ 0 & \Lambda^{(i)}_{(I+3,J+3)} \end{pmatrix}\lambda^i\right]\phi = \mathbf{D}(\lambda)\phi = 0 \tag{23}$$

where

$$\phi' = (\bar{U}_c - j\bar{U}_s, \bar{V}_c - j\bar{V}_s, \bar{W}_c - j\bar{W}_s, \bar{U}_c + j\bar{U}_s, \bar{V}_c + j\bar{V}_s, \bar{W}_c + j\bar{W}_s) \tag{24}$$

and ($\Lambda^{(i)}_{(I,J)}$, $\Lambda^{(i)}_{(I+3,J+3)}$; $I, J = 1, 2, 3$; $i = 0, 1, \ldots, 4$) are complex third order square matrices.

Due to the form of the pencil of (23) it is clearly seen that

$$\det\{D(\lambda)\} \rightarrow \begin{cases} \det\left\{\sum_{i=0}^{4} [\Lambda^{(i)}_{(I,J)}]\lambda^i\right\} = 0 & \text{(25a)} \\ \det\left\{\sum_{i=0}^{4} [\Lambda^{(i)}_{(I+3,J+3)}]\lambda^i\right\} = 0. & \text{(25b)} \end{cases}$$

Expanding equations (25) yields the required relations for λ, that is

$$D_1\lambda^8 \pm jD_2\lambda^7 + D_3\lambda^6 \pm jD_4\lambda^5 + D_5\lambda^4 \pm jD_6\lambda^3 + D_7\lambda^2 \pm jD_8\lambda + D_9 = 0 \tag{26}$$

where (D_i, $i = 1, 2, \ldots, 9$) are real quantities. Setting the anisotropic elastic compliances $E_{13}^{(0,1)}$ to zero, reduces (26) to its usual form for orthotropic and isotropic cylinders, that is,

$$\sum_{i=0}^{4} [D_{2i+1}\lambda^{8-2i}] = 0. \tag{27}$$

As (26) is complex, λ can take on the following values

$$\lambda_i = [\mp\lambda_{R_i} + j\lambda_{Ii}], -[\mp\lambda_{R_i} + j\lambda_{Ii}] \tag{28}$$

where λ_{Ri} and λ_{Ii} are the real and imaginary parts of λ. With these values of λ, equations (20) can easily be constructed and the usual procedure of evaluating the buckling and frequency eigenvalues follows.

FINITE FREELY SUPPORTED ORTHOTROPIC AND INFINITE ANISOTROPIC CYLINDERS

For finite freely supported orthotropic and infinite anisotropic cylinders a formal solution can be obtained for even or odd $P_r(\theta)$. This is achieved by applying the following Fourier transformation to equations (3a)–(3c) i.e.

$$\begin{aligned}
&\int_{0,-\infty}^{L,\infty}\int_0^{2\pi}\begin{bmatrix} F_1(x,\theta,\beta,M) & 0 & 0 \\ 0 & F_2(x,\theta,\beta,M) & 0 \\ 0 & 0 & F_3(x,\theta,\beta,M)\end{bmatrix}\begin{bmatrix} L_1(u,\ldots,t) \\ L_2(u,\ldots,t) \\ L_3(u,\ldots,t)\end{bmatrix} \mathrm{d}\theta\,\mathrm{d}x \rightarrow \\
&\quad -a_1\beta^2[U_{cc}(\beta,M), U_{ss}(\beta,M)] + a_2\beta M[U_{ss}(\beta,M), -U_{cc}(\beta,M)] \\
&\quad + RE_{33}[k+k_p]M^2[-U_{cc}(\beta,M), U_{ss}(\beta,M)] + \ldots \\
&\quad -a_{12}M^3[W_{sc}(\beta,M), W_{cs}(\beta,M)] + \rho R^2 h\omega^2[U_{cc}(\beta,M), U_{ss}(\beta,M)]\ldots = 0 \\
&\quad \vdots
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
F_1(x,\theta,\beta,M) &= (\cos M\theta\cos\beta x, \sin M\theta\sin\beta x) \\
F_2(x,\theta,\beta,M) &= (\sin M\theta\sin\beta x, \cos M\theta\cos\beta x) \\
F_3(x,\theta,\beta,M) &= (\cos M\theta\sin\beta x, \sin M\theta\cos\beta x)
\end{aligned}$$

with the first and second subscripts of u, v and w denoting the x and θ transform types, respectively. From the nature of the x-transform, equations (29) apply to both finite freely supported orthotropic and ∞ anisotropic cylinders, where β is treated either as a discreet $(N\pi/L, N = 0, 1, 2, \ldots)$ or continuous variable, respectively.

With the successive applications of these transforms, the shell displacement stability equations have been mapped into an infinite set of linear algebraic equations. To obtain the required equations for the case of uniform external pressures, ζ_γ and Γ_γ are set to zero for all $\gamma > 0$. This yields

$$[H_{(I,J)}]\psi + P_0[\mathbf{H}_{(I,J)}]\psi - \rho\omega^2 hR^2[I]\psi = 0 \tag{30}$$

where $[H_{(I,J)}]$ and $[\mathbf{H}_{(I,J)}]$ are defined in Appendix 2, $[I]$ is the usual identity matrix and the transpose of ψ is given by

$$\psi' = (U_{cc}, V_{ss}, W_{sc}, U_{ss}, V_{cc}, W_{cs}). \tag{31}$$

In matrix form equations (29) are written as

$$[\Xi]_\psi = 0 \tag{32}$$

where the ∞ matrix pencil $[\Xi]$ and the transpose of its associated latent vector are

$$[\Xi] = [H_{(I,J)}] + P_0[A_{(I,J)} + \mathbf{H}_{(I,J)}] + \Omega[I] \tag{33}$$

$$\psi' = \{(U_{cc}(\beta, 0), \ldots, W_{cs}(\beta, 0)), \ldots, (U_{cc}(\beta, k), \ldots, W_{cs}(\beta, k)), \ldots\} \tag{34}$$

with $A_{(I,J)}$ and Ω being defined in Appendix 3.

The question of convergence of equations (32) to the required solution cannot be answered in general for arbitrary functions $Pr(\theta)$. For this reason a criterion will be given from which convergence can be ascertained for a particular given $Pr(\theta)$. From the previous discussion it is easily seen that the problem of convergence of the infinite set (32) reduces to the requirements for the convergence of an infinite determinant. To assure convergence of the det. $[\Xi]$ a sufficient condition is the absolute convergence of the infinite series

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \alpha_{ij} < \infty \tag{35}$$

where α_{ij} are the elements of $[\Xi]$. Although this criterion is over restrictive, it is adequate for the present analysis. From the nature of (35) it is noted that point-wise equivalence throughout the whole domain of θ is not required of equation (5) for convergence of P_0 to significant decimal accuracy.

Since equation (32) has an infinite pencil, in order to obtain a solution, $[\Xi]$ is truncated to finite size. From the form of $[\Xi]$ it is clearly seen that the truncated matrix $[A_{(I,J)}]$ is nonsingular, thus the usual stability and frequency eigenvalue problems can easily be obtained. These are given by

$$\left\{\frac{1}{P_0}[I] + [H_{(I,J)}]^{-1}[A_{(I,J)} + \mathbf{H}_{(I,J)}]\right\}\psi = 0 \tag{36}$$

and

$$\{\Omega[I] + [H_{(I,J)}] + P_0[A_{(I,J)} + \mathbf{H}_{(I,J)}]\}\psi = 0 \tag{37}$$

where $[H_{(I,J)}]^{-1}$ is

$$[H_{(I,J)}]^{-1} = \begin{vmatrix} [\;\;]^{-1} & & 0 \\ & [\;\;]^{-1} & \\ 0 & & \ddots \end{vmatrix}. \tag{38}$$

For orthotropic cylinders the foregoing solution can be extended to include loadings of the form $P_r(x, \theta)$, such that

$$P_r(x, \theta) = \sum_{\alpha=0}^{\infty} \sum_{\gamma=0}^{\infty} \{\zeta_{\gamma\alpha} \cos \gamma\theta\} \cos \frac{\alpha\pi x}{L}. \tag{39}$$

Under such loadings, (29) remains the same with the exception that terms associated with $Pr(x, \theta)$ are transformed as

$$\int_0^L \int_0^{2\pi} u_{,xx} \cos \frac{\alpha\pi x}{L} \cos \gamma\theta \cos \beta x \cos M\theta \, d\theta \, dx \rightarrow \left\{ -\left[\left(\beta + \frac{\alpha\pi}{L}\right)\right]^2 U_{cc}\left(\beta + \alpha\frac{\pi}{L}, M+\gamma\right) - \left[\left(\beta - \frac{\alpha\pi}{L}\right)\right]^2 U_{cc}\left(\beta - \alpha\frac{\pi}{L}, M+\gamma\right) - \left[\left(\beta + \frac{\alpha\pi}{L}\right)\right]^2 U_{cc}\left(\beta + \alpha\frac{\pi}{L}, M-\gamma\right) - \left[\left(\beta - \frac{\alpha\pi}{L}\right)\right]^2 U_{cc}\left(\beta - \alpha\frac{\pi}{L}, M-\gamma\right) \right\},$$

$$\int_0^L \int_0^{2\pi} u_{,xx} \cos \frac{\alpha\pi x}{L} \cos \gamma\theta \cos \beta x \cos M\theta \, d\theta \, dx \rightarrow \left\{ -\left[\beta + \frac{\alpha\pi}{L}\right]^2 U_{cc}\left(\beta + \frac{\alpha\pi}{L}, M+\gamma\right) - \left[\beta - \frac{\alpha\pi}{L}\right]^2 U_{cc}\left(\beta - \frac{\alpha\pi}{L}, M+\gamma\right) - \left[\beta + \frac{\alpha\pi}{L}\right]^2 U_{cc}\left(\beta + \frac{\alpha\pi}{L}, M-\gamma\right) - \left[\beta - \frac{\alpha\pi}{L}\right]^2 U_{cc}\left(\beta - \frac{\alpha\pi}{L}, M-\gamma\right) \right\} \tag{40}$$

where the functional dependence of $U_{cc}(\beta+\alpha, M+\gamma), \ldots,$ with respect to β, M, α and γ is given by

$$U_{cc}(\beta \pm \alpha, M \pm \gamma) = \int_0^L \int_0^{2\pi} \cos(\beta \pm \alpha)\frac{\pi x}{L} \cos(M \pm \gamma)\theta u_{(x,\theta)} \, d\theta \, dx$$
$$V_{ss}(\beta \pm \alpha, M \pm \gamma) = \int_0^L \int_0^{2\pi} \sin(\beta \pm \alpha)\frac{\pi x}{L} \sin(M \pm \gamma)\theta v_{(x,\theta)} \, d\theta \, dx$$
$$W_{sc}(\beta \pm \alpha, M \pm \gamma) = \int_0^L \int_0^{2\pi} \sin(\beta \pm \alpha)\frac{\pi x}{L} \cos(M \pm \gamma)\theta w_{(x,\theta)} \, d\theta \, dx \tag{41}$$

Since for the orthotropic case, $E^{(0,1)}_{\substack{13\\23}} = 0$, equations (29) uncouple and therefore only the first set consisting of dependent variables U_{cc}, V_{ss} and W_{sc} need be solved.

DISCUSSION AND NUMERICAL EXAMPLES

With the aid of the exponential Fourier transform, solutions to nonuniformly prestressed macroscopically anisotropic cylinders have been developed herein. As mentioned earlier, the procedure outlined by equations (6) and (7) can, in a straight manner, be applied to general shells of revolution to evaluate buckling and frequency eigenvalues. Furthermore the solutions can also be extended to incorporate circumferential variations in shell geometry and material properties.

Using polynomial matrices in conjunction with Forsberg's method, an "exact" solution has been developed for the special case of anisotropic cylinders with lateral prestresses of the form $Pr(\theta)$. For such loadings, the solution is general, as all categories of shell boundary conditions can be handled.

In general all the solutions developed herein have reduced to linear eigenvalue problems of unbanded non-Hermitian matrices. Compared to the Hermitian case, the calculation of the eigenvalues of non-Hermitian matrices is less understood. A vital aspect of the calculation is the stability with respect to the growth of roundoff errors. For the present study the pencils of the matrix equations are reduced to Hessenberg form. The eigenvalues of these reduced matrices are subsequently obtained through the use of the numerically stable QR transformation [15].

As an initial numerical example, the stability of freely supported cylinders with loadings of the form $Pr(\theta) = P_0(1+\varepsilon\cos\gamma\theta)$ are studied in detail. Figures 1–3 show the effects on the shell stability of independently varying L/R, ε and γ. As expected, the allowable P_0 decreases monotonically with ε and/or L/R. Furthermore from Fig. 3 it is clearly seen that for loadings of the type described above, the $\gamma = 1$ case causes the greatest effects on the allowable P_0.

Recently Kalnins [3] noted that even though a nonsymmetric prestress may have to be expanded in a Fourier series with many terms, the components which affect the stability of the whole will be those with wave numbers $\gamma = 0$ and $\gamma = 1$. This statement must be qualified in the sense that it is true only for rapidly convergent series whose $\gamma = 0, 1$ terms predominant. For example Fig. 3 clearly shows that the effects of the $\gamma > 1$ terms can be significant. A further example is the stability of a freely supported cylinder with a band of uniform lateral pressure defined by

$$P_r(\theta) = \begin{cases} P_0, |\theta| < \Delta \\ 0, \Delta \le \theta \le \pi; -\Delta \ge \theta \ge -\pi. \end{cases}$$

For $\Delta > 1$ the $\gamma = 0$ term is predominant. As $\Delta \to 0$ the importance of the higher order series terms increase as clearly demonstrated in Fig. 4.

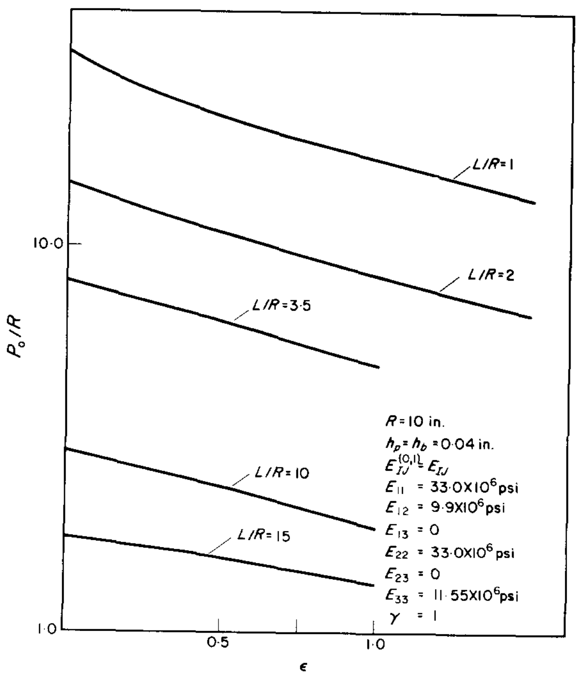

FIG. 1. Buckling eigenvalues for various L/R ratios.

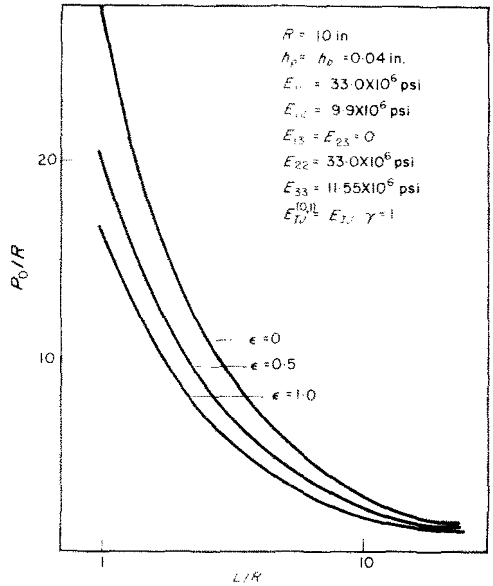

FIG. 2. Effects of ε on the buckling eigenvalue spectrum.

Examples of the effects of nonuniform lateral pressure on the frequency spectrum of freely supported cylinders are shown in Figs. 5 and 6. Here the lowest frequency branch decreases monotonically with ε and/or L/R. Extending the abcissa in Fig. 5 yields the critical value of P_0. Furthermore setting $\varepsilon \to 0$ yields the frequency eigenvalues of the uniform case. Figures 7 and 8 show the effects of anisotropy on the frequency spectrum of an

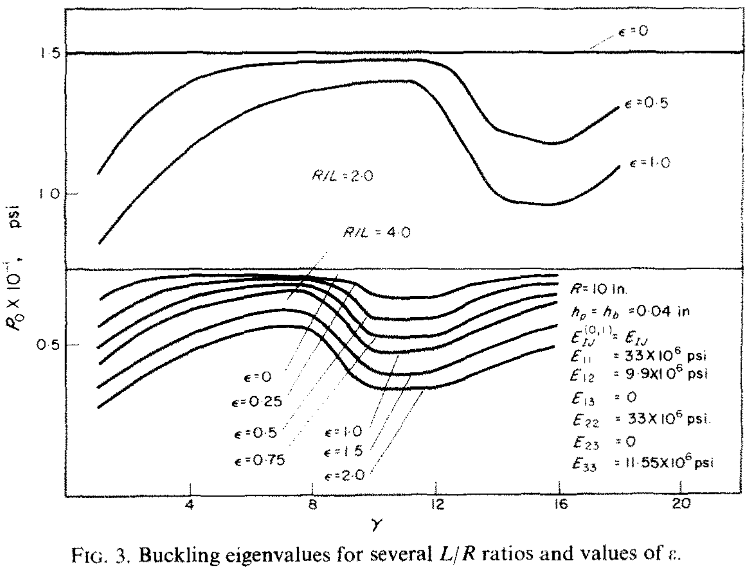

FIG. 3. Buckling eigenvalues for several L/R ratios and values of ε.

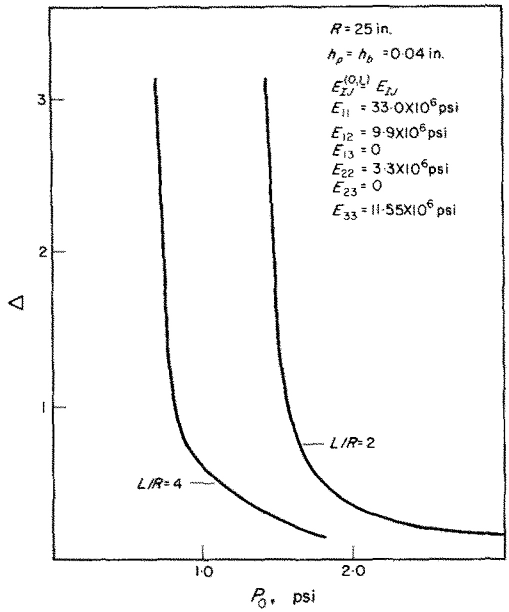

FIG. 4. Buckling eigenvalues of a cylinder with a band of lateral prestress in the axial direction.

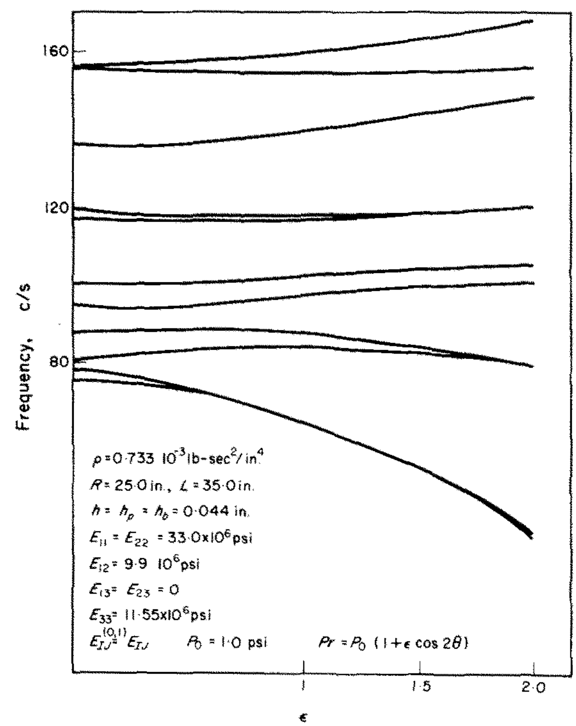

FIG. 5. Frequency eigenvalues for various values of ε.

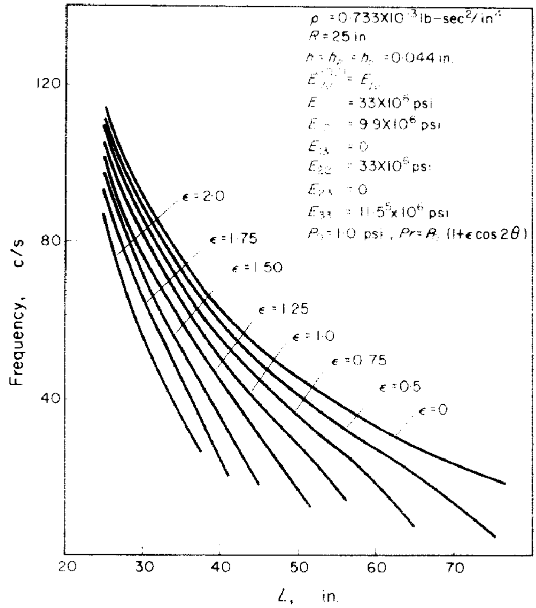

Fig. 6. Effects of ε on the frequency eigenvalue spectrum.

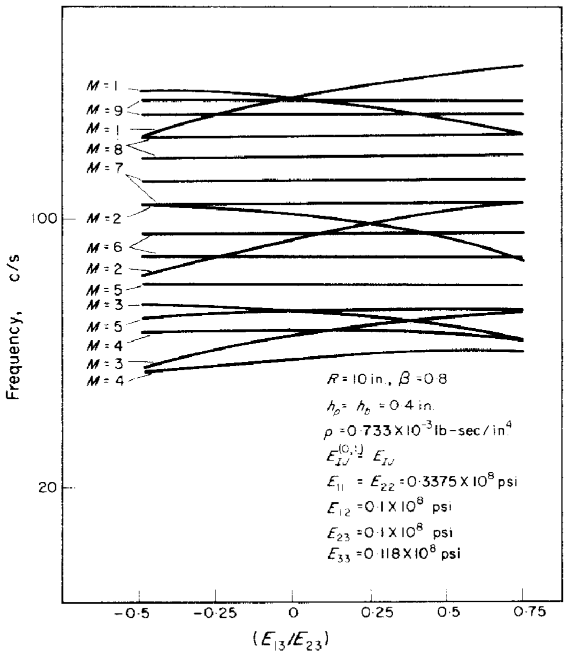

FIG. 7. Effects of E_{13}/E_{23} on the lowest eigenvalue branch of an infinite cylinder.

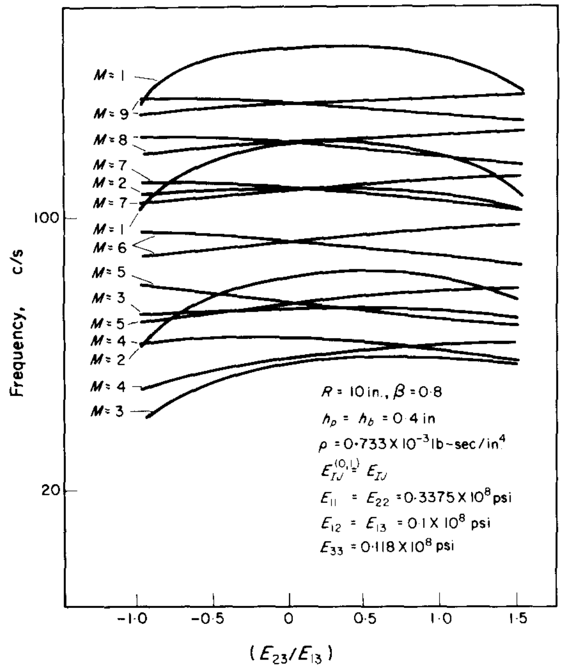

FIG. 8. Effects of E_{23}/E_{13} on the lowest eigenvalue branch of an infinite cylinder.

infinite cylinder. As is clearly seen, the introduction of anisotropy doubles the number of frequency branches for each value of M. From Fig. 7 it is seen that changes in the magnitude of E_{13} have lessening influence on the frequency as M increases.

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APPENDIX 1

$$a_1 = RE_{11}^{(0)}k - P_a$$
$$a_2 = 2[RkE_{13}^{(0)} - \bar{T}]$$
$$a_3 = R[E_{33}^{(0)}(k+k_p) - P_r]$$
$$a_4 = RE_{13}^{(0)}[k+k_p]$$
$$a_5 = Rk[E_{12}^{(0)} + E_{33}^{(0)}]$$
$$a_6 = RkE_{23}^{(0)}$$
$$a_7 = R[E_{12}^{(0)}k + P_r]$$
$$a_8 = RE_{23}^{(0)}[k+k_p]$$
$$a_9 = RE_{33}^{(0)}k_p$$
$$a_{10} = -RE_{13}^{(0)}k_p$$
$$a_{11} = -RE_{11}^{(0)}k_p$$
$$a_{12} = RE_{23}^{(0)}k_p$$
$$c_1 = -E_{11}^{(1)}k_b$$
$$c_2 = E_{23}^{(1)}k_b$$
$$c_3 = -E_{13}^{(1)}k_b$$
$$c_4 = E_{33}^{(1)}k_b$$
$$c_5 = R[E_{12}^{(0)}k + P_r]$$
$$c_6 = RE_{23}^{(0)}[k+k_p]$$
$$c_7 = -2E_{13}^{(1)}k_b$$
$$c_8 = -2E_{23}^{(1)}k_b$$
$$c_9 = -3E_{33}^{(1)}k_b - E_{12}^{(1)}k_b$$
$$c_{10} = RE_{23}^{(0)}k - 2\bar{T}$$

$$b_1 = RE_{13}^{(0)}k + E_{13}^{(1)}k_b$$
$$b_2 = RE_{23}^{(0)}[k+k_p] - E_{23}^{(1)}k_b$$
$$b_3 = R[E_{12}^{(0)} + E_{33}^{(0)}]k$$
$$b_4 = RE_{33}^{(0)}[k+k_p] + 2E_{33}^{(1)}k_b - P_a$$
$$b_5 = R[kE_{22}^{(0)} - P_r]$$
$$b_6 = 2[RkE_{23}^{(0)} + E_{23}^{(1)}k_b - \bar{T}]$$
$$b_7 = -Rk_pE_{13}^{(0)} - E_{13}^{(1)}k_b$$
$$b_8 = RE_{22}^{(0)}k_p - E_{22}^{(1)}k_b$$
$$b_9 = Rk_pE_{23}^{(0)} - 3E_{23}^{(1)}k_b$$
$$b_{10} = -RE_{33}^{(0)}k_p - k_b[2E_{33}^{(1)} + E_{12}^{(1)}]$$
$$b_{11} = RE_{23}^{(0)}k - 2\bar{T}$$
$$b_{12} = RE_{22}^{(0)}[k+k_p] - E_{22}^{(1)}k_b - P_rR$$
$$c_{11} = R[E_{22}^{(0)}k - P_r]$$
$$c_{12} = E_{11}^{(1)}k_b$$
$$c_{13} = E_{22}^{(1)}k_b$$
$$c_{14} = 2[E_{12}^{(1)}k_b + 2E_{33}^{(1)}k_b]$$
$$c_{15} = 4E_{23}^{(1)}k_b$$
$$c_{16} = 4E_{13}^{(1)}k_b$$
$$c_{17} = P_a$$
$$c_{18} = RE_{22}^{(0)}k_p + E_{22}^{(1)}k_b + RP_r$$
$$c_{19} = RE_{23}^{(0)}k_p + E_{23}^{(1)}k_b + 2\bar{T}$$
$$c_{20} = RE_{22}^{(0)}[k+k_p]$$

$$k = \frac{h_p}{R}, \qquad k_p = \frac{1}{12}\left(\frac{h_p}{R}\right)^3, \qquad k_b = \frac{h_b^3}{12R^2}.$$

APPENDIX 2

$$I, J = 1, 2, 3, \ldots$$
$$H(I,J) = \beta^2 a_1 + M^2 RE_{33}^{(0)}[k+k_p]$$
$$H(I,J+1) = -\beta M a_5$$
$$H(I,J+2) = -\beta RE_{12}^{(0)}k + \beta M^2 a_9 + \beta^3 a_{11}$$
$$H(I,J+3) = -\beta M a_2$$
$$H(I,J+4) = \beta^2 a_4 + M^2 a_6$$
$$H(I,J+5) = -Ma_8 + \beta^2 M a_{10} + M^3 a_{12}$$
$$H(I+1,J) = -\beta M b_3$$
$$H(I+1,J+1) = \beta^2 b_4 + M^2 RkE_{22}^{(0)}$$
$$H(I+1,J+2) = -M^3 b_8 - \beta^2 M b_{10} + M[RE_{22}^{(0)}(k+k_p) - E_{22}^{(1)}k_b]$$
$$H(I+1,J+3) = \beta^2 b_1 + M^2 b_2$$
$$H(I+1,J+4) = -\beta M b_6$$
$$H(I+1,J+5) = -\beta^3 b_7 - \beta M^2 b_9 + \beta b_{11}$$
$$H(I+2,J) = \beta^3 c_1 + \beta M^2 c_4 - \beta RE_{12}^{(0)}k$$
$$H(I+2,J+1) = -\beta^2 M c_9 + MRkE_{22}^{(0)}$$
$$H(I+2,J+2) = \beta^4 c_{12} + M^4 c_{13} + M^2\beta^2 c_{14} - \beta^2 c_{17} + c_{20} - M^2[RE_{22}^{(0)}k_p + E_{22}^{(1)}k_b]$$
$$H(I+2,J+3) = -M^3 c_2 - \beta^2 M c_3 + M c_6$$

$$H(I+2, J+4) = \beta^3 c_7 + \beta M^2 c_8 - \beta c_{10}$$
$$H(I+2, J+5) = \beta M^3 c_{15} - \beta M c_{19} + \beta^3 M c_{16}$$
$$H(I+3, J) = -\beta M a_2$$
$$H(I+3, J+1) = \beta^2 a_4 + M^2 a_6$$
$$H(I+3, J+2) = M a_8 - [\beta^2 M a_{10} + M^3 a_{12}]$$
$$H(I+3, J+3) = \beta^2 a_1 + M^2 R E_{33}^{(0)}[k + k_p]$$
$$H(I+3, J+4) = -\beta M a_5$$
$$H(I+3, J+5) = \beta R E_{12}^{(0)} k - \beta M^2 a_9 - \beta^3 a_{11}$$
$$H(I+4, J) = \beta^2 b_1 + M^2 b_2$$
$$H(I+4, J+1) = -\beta M b_6$$
$$H(I+4, J+2) = \beta^3 b_7 + \beta M^2 b_9 - \beta b_{11}$$
$$H(I+4, J+3) = -\beta M b_3$$
$$H(I+4, J+4) = \beta^2 b_4 + M^2 R k E_{22}^{(0)}$$
$$H(I+4, J+5) = M^3 b_8 + \beta^2 M b_{10} - M[R E_{22}^{(0)}(k + k_p) - E_{22}^{(1)} k_b]$$
$$H(I+5, J) = M^3 c_2 + \beta^2 M c_3 - M c_6$$
$$H(I+5, J+1) = -\beta^3 c_7 - \beta M^2 c_8 + \beta c_{10}$$
$$H(I+5, J+2) = \beta M^3 c_{15} + \beta^3 M c_{16} - \beta M c_{19}$$
$$H(I+5, J+3) = -\beta^3 c_1 - \beta M^2 c_4 + \beta R E_{12}^{(0)} k$$
$$H(I+5, J+4) = \beta^2 M c_9 - M R E_{22}^{(0)} k$$
$$H(I+5, J+5) = \beta^4 c_{12} + M^4 c_{13} + \beta^2 M^2 c_{14} - \beta^2 c_{17} + c_{20} - M^2[R E_{22}^{(0)} k_p + E_{22}^{(1)} k_b]$$

$$\mathbf{H}(I+i, J+j) = 0 \begin{cases} i = 0, 1, 2\,; j = 3, 4, 5 \\ i = 3, 4, 5\,; j = 0, 1, 2 \end{cases}$$

$$\mathbf{H}(I, J) = -M^2 R \qquad \mathbf{H}(I+3, J+3) = -M^2 R$$
$$\mathbf{H}(I, J+1) = 0 \qquad \mathbf{H}(I+3, J+4) = 0$$
$$\mathbf{H}(I, J+2) = -\beta R \qquad \mathbf{H}(I+3, J+5) = \beta R$$
$$\mathbf{H}(I+1, J) = 0 \qquad \mathbf{H}(I+4, J+3) = 0$$
$$\mathbf{H}(I+1, J+1) = -M^2 R \qquad \mathbf{H}(I+4, J+4) = -M^2 R$$
$$\mathbf{H}(I+1, J+2) = -MR \qquad \mathbf{H}(I+4, J+5) = MR$$
$$\mathbf{H}(I+2, J) = -\beta R \qquad \mathbf{H}(I+5, J+3) = \beta R$$
$$\mathbf{H}(I+2, J+1) = -MR \qquad \mathbf{H}(I+5, J+4) = MR$$
$$\mathbf{H}(I+2, J+2) = -M^2 R \qquad \mathbf{H}(I+5, J+5) = -M^2 R.$$

APPENDIX 3

$$[A(I, J)]\psi = \begin{bmatrix} A_1 \\ A_2 \\ \cdot \\ \vdots \\ A_l \\ \cdot \end{bmatrix}$$

$$A_l = \begin{bmatrix} \mathbf{A}_1 \\ \mathbf{A}_2 \\ \cdot \\ \vdots \\ \mathbf{A}_6 \end{bmatrix}$$

$$\mathbf{A}_1 = \frac{R}{2}\sum_{M=0}^{\infty}\sum_{\gamma=1}^{\infty}\{-(M-\gamma)^2U_{cc}(\beta, M-\gamma)-(M+\gamma)^2U_{cc}(\beta, M+\gamma)$$

$$-\beta W_{sc}(\beta, M-\gamma)-\beta W_{sc}(\beta, M+\gamma)\}\zeta_\gamma$$

$$\mathbf{A}_2 = \frac{R}{2}\sum_{M=0}^{\infty}\sum_{\gamma=1}^{\infty}\{-(M-\gamma)^2V_{ss}(\beta, M-\gamma)-(M+\gamma)^2V_{ss}(\beta, M+\gamma)$$

$$-(M-\gamma)W_{sc}(\beta, M-\gamma)-(M+\gamma)W_{sc}(\beta, M+\gamma)\}\zeta_\gamma.$$

$$\mathbf{A}_3 = \frac{R}{2}\sum_{M=0}^{\infty}\sum_{\gamma=1}^{\infty}\{-\beta U_{cc}(\beta, M-\gamma)-\beta U_{cc}(\beta, M+\gamma)-(M-\gamma)V_{ss}(\beta, M-\gamma)$$

$$-(M-\gamma)V_{ss}(\beta, M+\gamma)-(M-\gamma)^2W_{sc}(\beta, M-\gamma)$$

$$-(M+\gamma)^2W_{sc}(\beta, M+\gamma)\}\zeta_\gamma$$

$$\mathbf{A}_4 = \frac{R}{2}\sum_{M=0}^{\infty}\sum_{\gamma=1}^{\infty}\{-(M-\gamma)^2U_{ss}(\beta, M-\gamma)-(M+\gamma)^2U_{ss}(\beta, M+\gamma)+W_{cs}(\beta, M-\gamma)$$

$$+\beta W_{cs}(\beta, M+\gamma)\}\zeta_\gamma$$

$$\mathbf{A}_5 = \frac{R}{2}\sum_{M=0}^{\infty}\sum_{\gamma=1}^{\infty}\{-(M-\gamma)^2V_{cc}(\beta, M-\gamma)-(M+\gamma)^2V_{cc}(\beta, M+\gamma)$$

$$+(M-\gamma)W_{cs}(\beta, M-\gamma)+(M+\gamma)W_{cs}(\beta, M+\gamma)\}\zeta_\gamma$$

$$\mathbf{A}_6 = \frac{R}{2}\sum_{M=0}^{\infty}\sum_{\gamma=1}^{\infty}\{\beta U_{ss}(\beta, M-\gamma)+\beta U_{ss}(\beta, M+\gamma)+(M-\gamma)V_{cc}(\beta, M+\gamma)$$

$$+(M+\gamma)V_{cc}(\beta, M+\gamma)-(M-\gamma)^2W_{cs}(\beta, M-\gamma)$$

$$-(M+\gamma)^2W_{cs}(\beta, M+\gamma)\}\zeta_\gamma$$

$$\Omega = -\rho h\omega^2R^2.$$

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Абстракт—Даюься решения для определения собственных значений частоты и устойчивости макроскопически полно анизотропных круглых цилиндрических оболочек, подверженных действию неоднородного горизонтального предварительного напряжения. В анализе учитывается также наличие крутильного предваритеьного напряжения. Полученные результаты применимы к каждому типу предварительного напряжения, который удовлетворяет условиям Дирихле для рядов Фурье. Далее, пользуясь способом представлённым в работе на основе метода Кальнинса, можно получить подобные результаты, для общих оболочек вращения.